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Convection of Viscous Fluids: Energetics, Self-Similarity,
Experiments, Geophysical Applications and Analogies

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ABSTRACT

Main results of this paper are the formulas for the mean convection velocities, $\bar{u}$, of a viscous fluid and for the mean temperature difference in the bulk of the convecting fluid. These have been obtained by several ways: by scaling analysis of the Boussinesq equations, by analysis of the energetics of the process, and by using similarity and dimensional arguments. The last approach defines the criteria of similarity and allows the proposition of some self-similarity hypotheses. By several simple new ways an expression for the efficiency coefficient γ of the thermal convection in transforming the heat supplied to the fluid into the rate of generation of kinetic energy of convection has also been obtained. This expression coincides with the one obtained earlier by Lloboutry (1972) and Hewitt et al. (1975), though these authors gave different meaning to this coefficient. An analogy is pointed out between non-turbulent convection of a viscous fluid and the structure of turbulence for scales less than Kolmogorov's internal viscous microscale of turbulence. The formulas obtained for $\bar{u}$ and γ agree quantitatively well with the results of numerical calculations of convection in the Earth's upper mantle (McKenzie et al., 1974; Hewitt et al., 1975; Houston and De Bremaecker, 1975) which were carried out for a broad range of values of the heat flux introduced into the mantle.

It is found that the formula for the convection mean velocity also describes well some results of numerical calculations by Williams (1967, 1971) who studied the flow structure in rotating annuli, where atmospheric general circulation was modelled. In those experiments the Reynolds number was of order 100. In order to clarify the limits of applicability of our formula for $\bar{u}$ special simple experiments have been performed. In these experiments, I checked the proposed theoretical linear dependence of $\bar{u}$ on the depth of convecting fluid. The measurements revealed that this dependence is observed with a fair accuracy at least up to $Re \sim 10^3$.

Next, a simple phenomenological theory of gravitational convection is developed when motions arise due to density differentiation on the lower boundary of fluid or within it. It is shown that this case and the thermal case can be described in a unified way by introducing the density deficit flux. Applications of the theories to mantle convection are discussed rather briefly. The observed value of the geothermal flux gives a limitation to the intensity of gravitational convection and to the rate of differentiation of the mantle material. A simple experiment has been devised and its results confirm some basic conclusions of this elementary theory. Applications of these theories to mantle convection are discussed briefly.

In the last section an attempt is made to classify some forced flows. The two types of convection described here and turbulence beyond the Kolmogorov microscale are one type of flow. For these kinds of flows the total kinetic energy does not depend on the mass of the fluid in motion. The last property also exists in the theory of similarity for the circulation of the planetary atmospheres (Golitsyn, 1970, 1973). The turbulent convection, in the boundary layer, or in the fixed volume (described in Appendix 3), some models of general circulation of a non-rotating atmosphere (Leovy and Pollack, 1973; Burangulov and Zilitinkevich, 1976) and the turbulence in the inertial range of scales form another type of flows which could be called inertial or turbulence dominated. All of these aforementioned flows (or flow models) have one common feature, obeying "the rule of the fastest response," which says that the total kinetic energy of the fluid system is equal to the total supplied power multiplied by the shortest relaxation time of the system. The rule, found empirically by the author, is not a universal one but it may have some pragmatic value.

Appendix 1 is devoted to a discussion of the importance of the convection efficiency coefficient. In Appendix 2 the connection of the elementary theory developed here with the usual description of convection in particular in terms of the Rayleigh

number, Ra . It is shown that for large Ra the temperature field is almost isothermal within the bulk of fluid and the main changes are within the thermal boundary layers.

INTRODUCTION

Convection of a viscous fluid is discussed in thousands of papers, dozens of books and reviews and nevertheless, due to the nonlinearity of the phenomenon, the problem is still far from a complete solution and each new setting of the problem requires a new analytic, numerical or experimental approach. Theoretically and experimentally convection has mainly been studied by prescribing the temperature difference at the surfaces bounding the fluid. Much fewer are cases when the heat flux was given, as is the case for most natural phenomena where convection is observed. The mere change in the setting of the problem allows one to obtain by elementary ways a number of sufficiently general and useful results described here.

This paper originated from my old interest in the problem of motions in the Earth's upper mantle and only quite recently I realized its more general meaning. Due to existing large uncertainties in our knowledge on the mantle properties, in the details and even in the causes of the convection in the mantle my goal was only to obtain simple estimates of mean velocities in dependence upon the determining parameters and by simple but general ways.

First preliminary estimates of velocities within the mantle were obtained in 1970 (in an unpublished paper) using some

"principles" noted by me in the theory of similarity for circulation of planetary atmospheres (Golitsyn, 1970, 1973). However, due to some reasons which became clear to me only recently, those estimates gave velocities an order of magnitude larger than the observed several centimeters per year -- velocities of lithospheric plates. An important paper by McKenzie et al. (1974) which I learned of only in the fall of 1976 due to R. Hide, served as an impulse to turn back to the problem. In the paper a detailed account of plate tectonics is presented (see also Turcotte, 1975), some estimates of convection characteristics are given on the basis of boundary layer arguments and, which is most impressive, the results of numerous numerical experiments are described in modelling convection in the upper mantle. Impressed by this paper I developed a similarity theory for the convection in a very viscous fluid which gave by dimensional arguments, formulas which agreed well with the numerical results by McKenzie et al. (1974). This stage of work was described in my short note (see Golitsyn, 1977a). While this note was in press, I wrote a letter to D. P. McKenzie who kindly sent me reprints of many of his papers. The most useful for me was the paper by Hewitt, McKenzie and Weiss (1975) on dissipative heating in convective flows. The results of the paper allowed me to check some conclusions of my first note and presented the possibility to understand the energetics of many other types of convective flows, in particular, of experiments in rotating

annuli on modelling atmospheric general circulation (this particular problem is considered in Golitsyn, 1977b). All this has allowed me to develop a rather general approach to the problem of viscous fluid convection which has a broader applicability than to flows with small Reynolds number and very large Prandtl number as it had been thought at first (Golitsyn, 1977a). As a by-product I noted some general features and similarities for a number of forced flows.

The present paper is constructed in the following way. First by the scaling analysis of the convection equations the estimate of the mean velocity:

$$u \sim \left(\frac{\alpha g f}{\mu c_p} \right)^{\frac{1}{2}} d \quad [1]$$

is obtained for a cell with the size d . Here α is the thermal expansion coefficient, g is the gravity acceleration, f is the density of the heat flux, μ is the dynamic viscosity, and c_p is the specific heat at constant pressure of the fluid. The analysis also gives a mean temperature difference in the bulk of the convecting fluid:

$$\delta T \sim \frac{1}{d} \left(\frac{f \nu}{\alpha g \rho c_p} \right)^{\frac{1}{2}}, \quad [2]$$

where $\nu = \mu/\rho$ is the kinematic viscosity and ρ is the density of the

fluid. Then an estimate is given of the efficiency coefficient γ of convection in transforming the supplied heat into the rate of generation of kinetic energy which -- the rate -- in a steady state is equilibrated by the mean rate of dissipation. The estimate is

$$\gamma = \frac{d}{\mu} = \frac{\alpha g d}{c_p}. \quad [3]$$

These results were evidently first obtained by Lliboutry (1972) from different arguments and also by Hewitt et al. (1975) by analysis of energy and entropy balance (see also Golitsyn, 1977b). The other derivation of the formula for $\bar{u}$ if γ is known may be obtained from the expression for the rate of kinetic energy dissipation per unit mass of a viscous fluid. This derivation specifies the value of the numerical coefficient in the formula [1] and [2].

Then an analysis of the similarity criteria is carried out for the convection of a viscous fluid. It is found that if we scale velocities by [1] and temperature by [2] then the equations do not contain any numerical values for small Reynolds numbers (except in the boundary conditions) which means that such convective flows are self-similar at least in the bulk of fluid (if the one length scale is present). Another type of dimensional argument is also produces the formula [1].

An exact analogy is noted between the convection of a viscous fluid and the structure of a velocity field in the developed turbulence for scales less than the Kolmogorov viscous microscale. To make the analogy more convincing the Kolmogorov formulas are obtained also by dimensional arguments which require some slight reformulation of the problem. Then it is shown that the analogy is valid for any type of non-turbulent convection disregarding the value of the Prandtl number.

Then the results are compared with the data from various numerical experiments on modelling the mantle convection (McKenzie et al. 1974; Hewitt et al. 1975; Houston and De Bremaecker, 1975), and on modelling convection in annuli (Williams, 1967, 1971). In order to understand the applicability of the formula [1] to the convection with a rather high Reynolds number, a few simple experiments were carried out which show that the expression [1] is valid up to $Re \sim 10^3$ at least. Some discussion of this fact is presented.

Then gravitational convection is considered which takes place due to density differentiation of the material at the boundary or within the fluid. Such convection in many respects is found to be similar to the thermal case. Moreover, it is possible to describe both types of convection in a uniform way by introducing the flux of the density deficit. Simple experiments confirm some conclusions of the theory.

After that there are applications of the results to the Earth's upper mantle convection and questions of its modelling are discussed. In the last section an attempt to classify some geophysical forced flows is made and a "rule of the fastest response" is formulated.

Though some of the results (e.g., [3]) are known or practically known (e.g., [1] may be readily obtained from the formulas [47] and [53] of McKenzie et al., 1974), they are obtained here by another and direct way which clarifies their meaning and shows the limits of their applicability. Other theoretical and experimental results are new. The discussion of all of this material in a single paper seemed to me useful for the construction of a uniform view of various phenomena quite different at first sight.

2. Scaling analysis and energetics of convection.

Convection of an incompressible fluid is usually described by the Boussinesq equations:

$$\frac{\partial \tilde{v}}{\partial \tilde{t}} + (\tilde{v} \cdot \nabla) \tilde{v} = -\frac{1}{\rho} \nabla p + \alpha g T^0 + \nu \Delta \tilde{v} \quad [4]$$

$$\nabla \cdot \tilde{v} = 0 \quad [5]$$

$$\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = k \Delta T + \frac{1}{C_p} (q + \varepsilon), \quad [6]$$

$$\varepsilon = \nu \frac{\partial v_i}{\partial x_k} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) \quad [7]$$

In the momentum equation [4] $\vec{v}$ is velocity, p is pressure, ρ is density, T^0 is the temperature deviation from its equilibrium (adiabatic) distribution in the absence of convection. In the energy equation [6] q is the intensity of heat sources per unit mass and ε is the heat intensity due to friction. Boundary conditions must be added here of no-slip for velocities, the heat flux on the lower boundary and say, the temperature at the upper level (for the problem of mantle convection).

An analysis of this set of equations and of their applicability to the upper mantle convection has been given by McKenzie et al. (1974). When the mantle is further considered I shall use their values of the parameters: $\rho = 3.7 \cdot 10^3 \text{ kg/m}^3$, $C_p = 1.2 \cdot 10^3 \text{ J/kgK}$, $g = 10 \text{ m/sec}^2$, $\alpha = 2 \cdot 10^{-5} \text{ K}^{-1}$, $\nu = 2 \cdot 10^{17} \text{ m}^2/\text{sec}$, $k = 1.5 \cdot 10^{-6} \text{ m}^2/\text{sec}$. The mean value of the geothermal flux f is about $0.06 \text{ W/m}^2 = 60 \text{ erg/cm}^2\text{sec} = 1.4 \cdot 10^{-6} \text{ cal/cm}^2\text{sec}$. For the mantle conditions the left-hand side of equation [4] is absolutely insignificant because the Reynolds number for a layer depth of $d = 7 \cdot 10^5 \text{ m} = 700 \text{ km}$ and velocities of order $10^{-9} \text{ m/sec} = 3 \text{ cm/yr}$ is of order 10^{-20} .

For further analysis it is convenient to use instead of [4] the equation for vorticity $\omega = \text{curl } v$ and instead of temperature the enthalpy difference $e_1 = e - e_0 = c_p(T - T_0)$, where T_0 is the temperature of the upper boundary. Then instead of [4] and [6] we shall have for $Re \ll 1$

$$\nu \Delta \omega = H^{-1} \nabla e_1 \times n, \quad n = g/g \quad [8]$$

$$\partial e_1 / \partial t + v \nabla e_1 = k \Delta e_1 + g + \epsilon, \quad [9]$$

where $H = c_p / \alpha g \quad [10]$

is a quantity with the dimension of length. If the medium is a perfect gas then $g/c_p = (dT/dz)_{ad}$ is the adiabatic lapse rate, $\alpha = 1/T$ and $H = T/(dT/dz)$ and i.e., it is a depth of adiabatically stratified gas layer (for air $H \approx 30$ km). For other media H may be considered as a characteristic depth of the layer of material stratified by gravity field. For the upper mantle $H = 6000$ km. For water $\alpha = 2 \cdot 10^{-4} K^{-1}$, $c_p = 4.2 \cdot 10^3 J/kgK$ and $H = 2000$ km.

Further on for simplicity we shall consider convection with large Péclet numbers (though this requirement is not quite necessary -- see the end of §4) when the diffusional heat transfer is small as compared with the convection one. For this the Prandtl number of the fluid $Pr = \nu/k = Pe/Re$ should be large enough. For the mantle, e.g.,

$Pe \approx 500$ because $Pr \approx 10^{23}$. As a rule the contribution of viscous heating in eq. [9] is also small (conditions for its neglect will be discussed later; the special study of its role in this respect have been carried out by Hewitt et al., 1975). As a result in the stationary case the eq. [9] becomes

$$\nu \nabla_e^2 \approx q. \quad [9']$$

The total heat flux observed at the upper boundary is equal to

$$f = \rho q d + f_1. \quad [11]$$

Here for simplicity after McKenzie et al. (1974) we consider $q = \text{const}$, otherwise the difference between fluxes at two boundaries of the layer is $f - f_1 = \int \rho q dz$. It is convenient to introduce again after McKenzie et al. (1974)

$$\beta = \rho q d / f = \rho q d / (\rho q d + f_1), \quad [12]$$

the part of the heat flux produced within the layer to the total flux. For $\beta = 0$ the heat is introduced only at the lower boundary and for $\beta = 1$ the heat is produced only within the layer. The value of q could be estimated as

$$q = \beta f / \rho d \approx f / \rho d \quad [13]$$

The simplest estimate of the characteristic velocity of convective motions U can be obtained from the scale analysis of eqs. [8] and [9']. For the case of a bounded cavity with one characteristic length scale d (note that computations by McKenzie et al., 1974 and Hewitt et al., 1975 were done for the square region) we have from [9'] accounting for [13] that $\nabla e_1 \sim q/U \sim f/Ud$. Substituting this into [8], taking into account that $\Delta \omega \sim \omega/d^2 \sim U/d^3$ we may finally get

$$U \sim \left(\frac{qd^3}{\nu H} \right)^{1/2} \sim \left(\frac{fd^2}{\mu H} \right)^{1/2} = \left(\frac{\alpha g f}{\mu c_p} \right)^{1/2} d . \quad [1']$$

For the characteristic temperature difference we get from [9'] then

$$\delta T = \frac{\delta e}{c_p} \sim \left(\frac{f \nu}{\alpha g \rho c_p} \right)^{1/2} \frac{1}{d} . \quad [2]$$

Because [9'] disregards thermal boundary layers (term $k\Delta T$ is omitted) this value is evidently scaling the temperature difference within the bulk of fluid, i.e., outside the boundary layers. For special discussion of the temperature field in the convective fluid see Appendix 2.

From eq. (7) for the rate of energy dissipation $\epsilon \sim \nu U^2/d^2$. Substituting here U from [1'] we get

$$\epsilon \sim qd/H \sim (f/\rho d)(d/H) . \quad [14]$$

The quantity

$$\gamma = d/H = \alpha g d / c_p \quad [3]$$

can be named the efficiency coefficient of convection in transforming the supplied heat into the rate of generation of kinetic energy of convective motions. This rate of generation in the steady state must be equaled in average by the rate of the kinetic energy viscous dissipation.

We may neglect ϵ in the energy equation [9] if $\gamma \ll 1$. For the upper mantle $\gamma = 700 \text{ km} / 6000 \text{ km} = 0.117$. It is because of this low value that the computations by McKenzie and his colleagues with and without viscous heating produced practically the same results on the patterns and intensity of convection. Hewitt et al. (1975) have shown that integrally, as averaged in space and time the viscous dissipation is exactly equal to the work of pressure forces and this fact clarifies why viscous dissipation is equal in average to the rate of generation of kinetic energy of convection.

In the same paper by Hewitt et al. (1975) there is a derivation of the formula [3] for the efficiency γ based on consideration of the global balance of energy and entropy in the fluid. With some corrections insignificant for the final result such a derivation is reproduced also by Golitsyn (1977b). However, the formula [3] was

evidently first obtained by Lliboutry (1972). He used very simple arguments deserved for reproduction here. When a unit volume, which is lighter by $\rho' = \rho\alpha T'$ of its environment, is lifted on the height d then a potential energy $\rho\alpha T'gd$ is gained which is transformed into heat. An amount of heat carried by the volume during convection is $c_p\rho T'$ if the diffusion of heat is small. The efficiency of convection as a heat engine may be defined as a ration of these two quantities

$$\gamma = \frac{\rho\alpha T'gd}{c_p\rho T'} = \frac{\alpha gd}{c_p} = \frac{d}{H}.$$

This derivation is essentially correct though not all the necessary details have been included, because convection is not only a heat engine but also a motion engine. It shows that the proportionality coefficient between ϵ and $f/\rho d$ is exactly unity (if all the heat comes from below; if $\beta < 1$ then Hewitt et al., 1975, have shown that $\gamma = 1 - \beta/2$ for $g = \text{const}$). The other details are in a derivation similar to this one after our formula [41]. At the end of Section 4 it will be shown that the requirement $Pe \gg 1$ is not essential and the estimate $\gamma \approx d/H$ as well as eq. [1] for the mean velocity are valid at any Prandtl number if Re is not too large. (See also Appendix 3.)

If we take the expression for the efficiency γ as a known, e.g., from the above described considerations, then the formula for the

mean convection velocity can be determined with accuracy up to a multiplier of order unity by a second means. Suppose then in the expression [7] for the mean rate of energy dissipation ϵ all derivatives are of the same order. This is supported by inspection of graphs of the stream function in computations by McKenzie et al. (1974). The graphs have a shape of more or less concentric circles spaced about uniformly. Approximate $\partial v_i / \partial x_k$ as $2U/d$. For the plane case as in McKenzie et al. (1974) or for roll convection in the sum [7] there are eight terms, therefore $\epsilon \approx 32\nu U^2/d^2$. At the other hand due to [14] $\epsilon \approx \gamma f / \rho d = f / \rho H$ (note a rather unexpected fact that the dissipation rate is determined only by the heat flux and the fluid properties, but not by geometry). Equate these two expressions for ϵ and get for U :

$$U \approx \left(\frac{fd^2}{32\mu H} \right)^{1/2} = \left(\frac{\alpha g f}{32\mu c_p} \right)^{1/2} d . \quad [1']$$

If the motion is essentially three-dimensional than in the sum [7] there are 18 terms and for the same approximation of derivatives one has in the eqn. [1'] 72 instead of 32. Therefore all conditions being the same the intensity (velocity) of the 3D-convection is not more than $(72/32)^{1/2} = 1.5$ times less than for 2D-case.

This derivation of the mean convective velocity is quite general. It requires only that viscosity be controlling the flow structure,

i.e., the Reynolds number should not be too small and the convection should not be fully turbulent. The discussion of this statement will be continued at the end of Section 5.

3. Similarity theory for the convection of viscous fluid.

The similarity theory is a base for modelling of various phenomena. In several cases the formulation of some self-similarity hypotheses gives a possibility to get by dimensional arguments some non-trivial results. In our case all this allows one to derive sufficiently rigorously eq. [1] again and by this to determine the conditions for its validity and for modelling convection.

Let us formulate our problem again for variables $\omega = \nabla \times v$ and $e_1 = c_p(T - T_0)$ for a general case. Then we have

$$\frac{d\omega}{dt} - (\omega \nabla) v = - \frac{1}{H} [\nabla e_1 \times n] + \nu \Delta \omega \quad [15]$$

$$\frac{de_1}{dt} = k \Delta e_1 + q + \epsilon \quad [16]$$

with the zero velocity for all boundaries and $e_1 = 0$ at $z = d$ and

$$k \partial e_1 / \partial z = -f_1 / \rho \quad \text{at } z = 0. \quad [17]$$

Transform this system to a non-dimensional form by taking d as a length scale, $\delta e = (f\nu H/\rho d^2)^{1/2}$ from [2] as the enthalpy scale, eq. [1] as the velocity scale. Then the time scale is defined as

$$\tau = d/U = (\mu c_p / \alpha g f)^{1/2} = (\mu H / f)^{1/2} . \quad [18]$$

It is interesting that the characteristic time of convection, i.e., the fluid particle turnover time does not depend on the geometric scales but only on heat flux and fluid properties. The equations with all the dependent and independent variables being non-dimensional are:

$$\text{Re} [d\omega/dt - (\omega \nabla) v] = -[\nabla e_1 \times n] + \Delta \omega , \quad [15']$$

$$de_1/dt = \text{Pe}^{-1} \Delta e_1 + q + \gamma \varepsilon \quad [16']$$

$$\partial e_1 / \partial z = -\text{Pe} \text{ at } z = 0 \text{ and } e_1 = 0 \text{ at } z = 1 \quad [17']$$

Here $\text{Re} = Ud/\nu$ is the Reynolds number. It can be written noting [18] as

$$\text{Re} = d^2/\nu\tau = \tau_v/\tau$$

i.e., to represent as a ratio of the two relaxation times: of the

viscous time $\tau_v = d^2/\nu$ to the turnover time $\tau = d/U$. In the energy equation [16'] there are two similar criteria: first is the Péclet number $Pe = Ud/k$ and the second $\gamma = d/H$ is the convection efficiency.

An important class of convective motions is when all the similarity criteria are small or large compared to unity. For the upper mantle, e.g., $Re \approx 3 \cdot 10^{-21}$, $Pe \sim 10^3$, $\gamma \sim 10^{-1}$. For $Pe \gg 1$ the relative width of the thermal boundary layer is small and if we disregard its structure then we may assume self-similarity of convection in the main volume of the fluid, because eqs. [15'] and [16'] will contain none of the similarity criteria, i.e., all the terms left will be of the same order. This means that the non-dimensional scales of velocity and enthalpy fluctuation are of order unity, i.e., estimates [1] and [2] are strictly valid if $Re \ll 1$, $Pe \gg 1$ and $\gamma \ll 1$. At these conditions, moreover, one may expect the similarity of flow and temperature patterns of convection outside the thermal boundary layer. We shall see later in Sections 4 and 5 that for validity of [1] and [2] the first two conditions are not necessary, the estimates are valid in a much broader range of Reynolds and Péclet numbers though outside the above range one should not expect the similarity of flow patterns.

If beside the vertical scale d there is a horizontal scale L then all the values have to be dependent on the ratio $d/L = H$.

For similarity of convective flow patterns one needs the same values of H and self-similarity with respect to other criteria. The connections of these similarity criteria with Rayleigh numbers are discussed in Appendix 2.

The derivation of eqn. [1] using more traditional similarity and dimensional arguments may be found in Golitsyn (1977a). Note that there, due to different scaling of velocity and temperature, there is another similarity criterium in the momentum equation.

Still another derivation of eqn. [1] can be proposed if we note that our equation system for the case $Re \ll 1$, $\gamma \ll 1$ and $q = 0$ is invariant relative to a group of transformations (the idea of this derivation was suggested to me by G. I. Barenblatt). For this case eqns. [15]-[17] are simplified to

$$\nabla H \Delta \tilde{\omega} = \nabla e_1 \times \tilde{n} \quad [15'']$$

$$de_1/dt = k \Delta e_1 \quad [16'']$$

$$k \partial e_1 / \partial z = -f/\rho \quad \text{at } z = 0 \text{ and } e_1 = 0 \text{ at } z = d. \quad [17'']$$

The structure of the system allows one to consider the dimension of enthalpy as arbitrary, but then the dimensions of external

parameters will be as follows: $[b] = [vH] = [e]LT$, $[f/\rho] = [e]LT^{-1}$, $[d] = L$ and $[k] = L^2T^{-1}$. Here $[e]$ is the dimension of enthalpy, L and T are dimensions of length and time. Note that in eqn. [15"] there is the combination $vH = b$ only and because we neglect inertial terms and viscous heating then the kinematic viscosity ν separately does not enter.

From the three first external parameters one may construct the quantity with the dimension of velocity

$$V = (fd^2/\rho vH)^{\frac{1}{2}} = (\alpha g f/\mu c_p)^{\frac{1}{2}}d = (f/\mu H)^{\frac{1}{2}}d. \quad [1]$$

If we account for the fourth parameter, the thermodiffusivity, then we may construct from all the parameters one non-dimensional combination, which is $Pe = Vd/k = (f/\mu H)^{\frac{1}{2}}(d^2/k)$. If $Pe \gg 1$ then the concrete value of k is not essential and we may not account for it in the set of external determining parameters.

The possibility of an arbitrary choice of the enthalpy dimension can be obtained from the next observation (by G. I. Barenblatt). The system [15"]-[17"] does not change its form under transformation

$$e_1 \rightarrow \alpha e, \quad vH \rightarrow \alpha vH, \quad f/\rho \rightarrow \alpha f/\rho \quad [19]$$

where α is an arbitrary number. If we would consider as usual $[e] = L^2 T^{-2}$, then $[f/\rho] = L^3 T^{-3}$, $[vH] = L^3 T^{-1}$, $[d] = L$, $[k] = L^2 T^{-1}$ and from the three first parameters one may already construct a new non-dimensional criterium $\Pi = (f/\rho)^{1/2} (vH)^{-1/6} d$. Under transformation [19] this criterium is changing as $\Pi \rightarrow \alpha^{1/3} \Pi$, i.e., it is dependent on the choice of the value of α , in contradiction to the basic system [15"]-[17"]. Therefore, in our case we may consider the dimension of enthalpy as arbitrary with total justification.

For a volume with the characteristic size d one obtains the formula for the total kinetic energy of the convection using [1']:

$$E = \frac{1}{2} \rho d^3 v^2 = \frac{c^2}{2} \frac{f d^5}{vH} \quad [20]$$

where coefficient $c^2 \approx 1/32$ or $1/72$ and may also depend on the shape of the region.

The formula [3] for the efficiency coefficient $\gamma \approx d/H$ may be understood in the following way at least for the case of $Re \ll 1$.

Including dissipation means the appearance of a new dimensional parameter in the energy equation [16"], i.e., the kinematic viscosity ν itself. Adding it to the existing parameters $b = vH$, f/ρ and d gives a possibility to produce the non-dimensional similarity

criterion $\gamma_1 = \nu d/b = d/H$. With respect to dissipation we have a typical example of self-similarity of the second kind in the terminology of G. I. Barenblatt (1976). The mean convection velocity for a viscous fluid is self-similar with respect to Π , Re and γ_1 ; the last independence means that U does not depend on ν separately, only on the combination $b = \nu H$. If we take an analogous hypothesis for the dissipation rate ϵ then we could construct the quantity with the dimension of $\epsilon([\epsilon] = [e]T^{-1}$ from [16]) from the parameters $b = \nu H$, f/ρ and d only as $\epsilon = c_1 f/\rho d$, where c_1 would be some numerical constant. The neglect of viscous heating means the neglect of the similarity criterion $\gamma_1 = \nu d/b$. But this can be done in determining the dissipation from the dimensional arguments only if $c_1 = c_1(\gamma_1) \rightarrow \text{const}$ at $\gamma_1 \rightarrow 0$. Here we have, after Hewitt et al. (1975)

$$\lim_{\gamma_1 \rightarrow 0} c_1(\gamma_1) = \gamma_1 \left(1 - \frac{\beta}{2}\right), \quad [21]$$

where β is from [12]. The non-existence of a finite limit for a constant when some similarity criterium tends to zero (or infinity) is the property of the self-similarity of the second kind. The constant usually has a power dependence on the criterium. The character of this dependence can not be defined from dimensional arguments and some other arguments should be applied (here the expression [7] for ϵ has been used).

Before seeing the paper by Hewitt et al. (1975) I determined the value of $\gamma = \epsilon M/f$ for the upper mantle empirically (not suspecting the complicated situation just described) using the results of computations by McKenzie et al. (1974). In Figs. 18-20 of that paper there are graphs of the horizontal velocity component u at the free upper surface of the computational cell in dependence on the horizontal coordinate x and the stream functions for three types of the lower boundary conditions: $\beta = 0, \frac{1}{2}$ and 1. Assuming as above all derivatives in the sum [7] of the same order (no boundary layers for u) we may write $\epsilon \approx \rho v (\partial u / \partial x)^2$. Differentiating graphically $u(x)$ and taking squares of the derivatives we may estimate the value of ϵ . For $\beta = 0$ (all heat is from below) the calculations give $(\partial u / \partial x)^2 = 0.16 \text{ mm}^2/\text{km}^2\text{yr}^2 = 1.6 \cdot 10^{-30} \text{ sec}^{-2}$ because $1 \text{ km} = 10^6 \text{ mm}$ and $1 \text{ year} = 3.15 \cdot 10^7 \text{ sec}$. From here $\epsilon = 2.5 \cdot 10^{-12} \text{ m}^2 \text{ sec}^{-3} \text{ H (W/kg)}$. It needs to be compared with the mean geothermal flux $f = 6 \cdot 10^{-2} \text{ W m}^{-2}$. Then we get $\gamma = \epsilon/f = \epsilon M/f = 0.11$. The calculations by the formula [3] $\gamma = d/H$ give $\gamma = 0.117$. Hewitt et al. (1975) performed special computations of the value of γ using their computed velocity fields. These computations were done for the mean values of the fluxes through the upper boundary equal to 10^{-3} , 10^{-2} and 5.85 W m^{-2} . In their computations the value of γ for $\beta = 0$ was changing from 0.10 up to 0.115. This confirms that the convection efficiency does actually depend only very slightly on the heat flux.

For $\beta = \frac{1}{2}$ the analogous calculations using graphs of Fig. 19 by McKenzie et al. (1974) give $\gamma = 8\%$ and for $\beta = 1$ (Fig. 20 -- all heat from within) I got $\gamma \approx 6\%$. According to [21] we get $\gamma = 8.7\%$ and 5.8% correspondingly. The exact computations by Hewitt et al. (1975) of integral dissipation by velocity fields give changes of γ for these two cases from 7.3 up to 8.4% and from 4.6 up to 5.4% , correspondingly. As we see, the results of the two approximate methods and of one computational method of determining the convection efficiency are very close between each other.

Such a value of the convection efficiency coefficient for the mantle appeared at first rather high for me. For comparison, the similar efficiency for the Earth's atmosphere in transforming solar heat into wind kinetic energy is of order 1% , and for the martian atmosphere it is still an order of magnitude less (Golitsyn, 1973).

The numerical experiments by McKenzie et al. (1974) were carried out with variation of the heat flux value from 10^{-4} up to 10^{-1} Wm^{-2} . This gives the possibility of checking the dependence of velocity on f in the three order range of changes of f . The formulas [56], [61] and [62] by McKenzie et al. (1974) describe dependencies of computed maximal values of the horizontal velocity $\bar{u}$ on the total heat flux at the upper boundary.

$$\begin{aligned}
 \beta = 0 \quad \lg \bar{u} &= 0.50 \lg f + 1.91, \\
 \beta = \frac{1}{2} \quad \lg \bar{u} &= 0.49 \lg f + 1.80, \\
 \beta = 1 \quad \lg \bar{u} &= 0.54 \lg f + 1.85.
 \end{aligned}
 \tag{22}$$

Here $\lg = \log_{10}$ (as in all Russian literature), $\bar{u}$ is measured in mm/yr, $[f] = \text{Wm}^{-2}$. Assuming that the maximal velocity u depends on f as the rms velocity $\bar{u}$ (given by [1]) we see that the character of the power dependence of velocity on f as $f^{\frac{1}{2}}$ produced by various arguments is fulfilled practically exactly in the numerical experiments and does not depend on the way of the introduction of heat into a system in accordance with our derivations (say in Section 2).

If we set in eqn. [20] $c = [(1 - \beta/2)/32]^{\frac{1}{2}}$ as it follows from [21] then taking the logarithm of [20] and the numerical values of the parameters entering there we may get f in the same units that

$$\lg \bar{u} = 0.5 \lg f + 1.76 + 0.5 \lg(1 - \beta/2); \tag{23}$$

where $\lg = \log_{10}$.

Remembering that eqns. [22] describe the maximal convection velocity, and [23] relates to the rms velocities we see that our result is valid not only from the point of view of dependencies on the external parameters of the problem but it is quite good in determining the value of the numerical coefficient in the formula for the mean velocity.

The dependence similar to [22] may be also obtained using the results of numerical computations by Houston and DeBremaecker (1975) for their case of constant viscosity $\nu = 5 \cdot 10^{17} \text{m}^2 \text{sec}^{-1}$, density $\rho = 3.5 \cdot 10^3 \text{kg/m}^{-3}$ and $f = 5.94 \cdot 10^{-2} \text{Wm}^{-2}$, assuming again that the maximal velocity is determined by a formula of the type of [1']. The computations give $\hat{u} = 16 \text{mm/yr}$ at the distribution of heat sources which corresponds to $\beta = 0.79$. The substitution in [1'] of the values just given produces

$$\lg \hat{u} = 0.5 \lg f + 1.82 \quad [22']$$

which agree quite well with eqn. [22]. Some further discussion of the results by McKenzie et al. is in Appendix 2.

4. One useful analogy to convection of viscous fluid

To clarify the character of mean convection velocity dependence on the external parameters let us consider in a similar way one classical problem of hydrodynamics but with a slight change of its posting to make it and the convection problem alike. Let us consider the sufficiently developed locally homogeneous and isotropic turbulent fluid and a volume in it with a characteristic size d smaller than the Kolmogorov viscous microscale

$$\eta = \nu^{3/4} \epsilon^{-1/4} \quad [24]$$

For distances r less than η Kolmogorov (1941) determined the structure tensor of the velocity field fluctuation. He found that

$$D_{\mathcal{L}\mathcal{L}}(r) = 2D_{nn}(r) = 2 r^2/15, \quad [25]$$

where $D_{\mathcal{L}\mathcal{L}}$ and D_{nn} are the longitudinal and lateral structure functions, the mean squares of the differences of the corresponding velocity components taken at two points separated by the distance r . Kolmogorov obtained formulas for $D_{\mathcal{L}\mathcal{L}}$ and D_{nn} from the expression for the mean rate of kinetic energy dissipation (see [7]) taking into account the isotropy and smoothness of velocity gradients at distances $r < \eta$.

Now we shall obtain an expression analogous to [25] by another way not expanding the velocity field in Taylor series on r as Kolmogorov did. Let us consider what determines the kinetic energy E of the fluid volume with the size d relative to its nearest environment at distances also of order d . That means we shall compare the kinetic energies of two neighboring volumes with the sizes $d < \eta$. The relative kinetic energy of such a volume should depend on its size d , on the viscosity ν and on the total rate of the energy inflow to this volume.

$$Q = \epsilon M = \epsilon \rho d^3, \quad [Q] = ML^2T^{-3}. \quad [26]$$

Using d , v , and Q we can by this unique way construct the quantity with the dimension of the energy

$$E = c_2 Q \cdot d^2 / v = c_2 Q \tau_v , \quad [27]$$

where c_2 is some numerical constant. From here and [26] the value of the velocity square in this volume is

$$U^2 = \frac{2E}{M} = \frac{2c_2 Q}{v \rho d} = 2c_2 \frac{\epsilon}{v} d^2 . \quad [28]$$

The identity of the expressions [25] and [28] is obvious but the last one is obtained by dimensional arguments similar to the ones used in the preceding section. The structure of these formulas is quite analogous to the structure of the formula [1'] for the mean velocity of convection of viscous fluid in a bounded volume with the size d if one remembers that in the convection $\epsilon \sim f/\rho H$ (see e.g., [14]). Then the square of the velocity scale from [1] can be written as

$$U^2 \sim \frac{\epsilon}{v} d^2 .$$

Up to now in order to obtain formulas for the mean velocity and efficiency of convection we have assumed $Re \ll 1$ and $Pe \gg 1$, so

$Pr > Re/Re \gg 1$. The two last conditions are not essential. After the just found analogy it is clear that any laminar convection in the sense of its intensity should be similar to the turbulent microstructure. In order to show this formally, write the energy equation [16] as

$$\frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} = -c_p \frac{\partial f'_i}{\partial x_i} + \varepsilon$$

Here

$$f'_i = \frac{f_i}{\rho c_p} = - \frac{k}{\rho c_p} \frac{\partial e}{\partial x_i} - \int \frac{q}{c_p} dz$$

is the kinematic heat flux, where for simplicity the distribution density of the volume heat sources q is supposed to be homogeneous in the horizontal. Assuming ε to be small and the scale of the heat flux known independently of its origin we get from here and from [15'] the same estimates [1] and [2] for the mean velocity $\bar{u}$ and the temperature difference* δT .

*For the purely diffusive flux $f' = k \partial T / \partial z \approx -k \delta T / d$ from [2] we may get quite naturally $\delta T \approx \frac{kv}{\alpha g d^3} Ra$, where Ra is the usual Rayleigh number.

Second, as it follows from the results of experiments described in the next section, the condition $Re \gg 1$ is also not necessary for the validity of the estimate [1]. It appears that the Reynolds number related in the convective conditions to the Rayleigh and Prandtl numbers (see Appendix 2) should not be too large in order that the convection become not fully turbulent (see Appendix 3). Up to the appearance of this regime the analogy between intensities of the turbulence viscous microstructure and convection of a viscous fluid holds up.

5. Various experiments on modelling the convection in the upper mantle and in other geophysical phenomena

It has already been noted that due to self-similarity of the convection, for the modelling of the mantle convection it is necessary to have $Re \ll 1$, and $Pe \gg 1$. The condition $\gamma \ll 1$ is always held in the laboratory. It is also necessary to have a fluid with $Pr = \nu/k \gg 1$ large enough. As is shown in Appendix 2, the Reynolds and Péclet number are related to the Rayleigh number and for setting up the convection the last number must exceed its critical value. Therefore, in a model it is not necessary to try to reach exact coincidence of the corresponding similarity criteria for the model and the mantle though the very fact of the self-similarity and the limits of its existence should be specially checked in the laboratory.

Let us consider from this point of view the laboratory experiments by Booker (1976). For the mantle convection an important factor is the dependence of viscosity on depth connected mainly with the dependence of viscosity on temperature (see, e.g., Carter, 1976). Therefore Booker chose as the working liquid a special kind of oil with the dynamic viscosity changing from 200 P (poise) at -20°C up to 0.2 P at $+80^{\circ}\text{C}$.

The Rayleigh number Booker defined for the mean temperature of the liquid $\bar{T} = \frac{1}{2}(T_1 + T_2)$ where T_1 and T_2 are temperatures of the lower warm and of the upper cold surfaces. The structure of the convective roll cells depended little on the variation of the liquid parameters with height. Using temperature dependencies for the parameters written in the Booker paper, we may find for $+30^{\circ}\text{C}$ that $\bar{\nu} = 2.2 \cdot 10^{-4} \text{m}^2 \text{sec}^{-1}$, $\bar{\alpha} = 5.7 \cdot 10^{-4} \text{K}^{-1}$, $\bar{k} = 7 \cdot 10^{-8} \text{m}^2 \text{sec}^{-1}$, $c_p = 2.1 \cdot 10^3 \text{ /kgK}$, $\bar{\rho} = 846 \text{ kg/m}^3$. The height of the cell was in the limits 1.43 - 1.50 cm. Then the Rayleigh number $Ra = \alpha g \Delta T d^3 / \bar{\nu} \bar{\rho} \approx 10^5$. The Nusselt number in his measurements was found to be $Nu = 0.184 Ra^{0.28}$, wherefrom $Nu \approx 4.6$.

Now by formula

$$f = k \rho c_p \Delta T d^{-1} Nu \quad [30]$$

we may calculate the heat flux into fluid. For $\Delta T = 100 \text{ K}$ we get

$f = 0.4 \text{ Wcm}^{-2}$. Having this value we determine the mean convection velocity from [1'] wherefrom $V \approx 0.6 \text{ mm/sec}$. Though Booker does not report the convective velocity estimates this value seems reasonable (from my limited experience of observing convective flows). For his experiments now we find $Re = 0.04$, $Pe = 120$, and $Pr = 3000$, and $\gamma = 4 \cdot 10^{-8}$.

Therefore these experiments satisfy all the conditions of the self-similarity for convection. The very large difference in the value of the similarity criterion for the dissipation (generation) of the kinetic energy $\gamma_1 = d/H$ is also not important. The experience of the laboratory and numerical modelling of the atmospheric general circulation in annuli (see, e.g., Hide and Mason, 1975, Dolzhansky and Golitsyn, 1977, Williams, 1967, 1971, Golitsyn 1977b) and some other examples show that the flow pattern does not much depend on the value of this criterion. It determines only the intensity of the flow. Therefore the experiments of the Booker type allow in principle the extraction of much qualitative and even quantitative information of the flows in the Earth's upper mantle. The first goal in such experiments should be the check of the dependence $V \sim f^{\frac{1}{2}}$ or of the whole dependence [1] and [3].

Quite surprising was the fact that the dependence of the mean convection velocity on the external parameters similar to [1] is

also valid for motions in bounded vessels when the Reynolds number composed by their rms velocity and the characteristic vessel size is not small but has the value of order several tens or even hundreds. This had been found while analyzing the energetics of convection of water in rotating annuli using the data of detailed computations of the process by Williams (1967, 1971). Though the heating and cooling from lateral boundaries and rotation decrease the efficiency several times in comparison with the heating from below of the non-rotating fluid (the situation has been analysed in some detail by Golitsyn, 1977b), nevertheless if we know the specific dissipation ϵ the mean velocity can be well estimated by the formula

$$\bar{u} \approx (\epsilon/32\nu)^{\frac{1}{2}} d \quad [31]$$

for the axisymmetric case of convection (Williams, 1967) and by

$$\bar{u} \approx (\epsilon/72\nu)^{\frac{1}{2}} d \quad [32]$$

for the three-dimensional case when baroclinic waves are developing in an annulus (Williams, 1971). Note the similarity of these formulas with Kolmogorov's formulas [25] or [28].

In fact, using the results of the direct computations by Williams (1967) one can find for his case A3 the rms velocity

$V = 2.6$ mm/sec. The dissipation ϵ can also be found from his results $\epsilon = 3 \cdot 10^{-3}$ cm²/sec⁻³. Using now the depth of his vessel $d = 5$ cm one obtains using [31] $V = 2.4$ mm/sec. For the baroclinic case the analogous estimate by [32] is 1.1 mm/sec compared with the computed value of 1.2 mm/sec ($\epsilon = 1.1 \cdot 10^{-3}$ cm²sec³, $d = 3$ cm; for details see Golitsyn, 1977b).

The Reynolds numbers are for the two cases of order 100 and 40, correspondingly. One may say that these values are on the one hand sufficiently large in order that boundary layers are relatively thin but on the other hand the values are still small in the sense that the flow pattern is still laminar though rather complicated, i.e., the viscosity forces determine essentially the flow patterns and the velocity gradients are appreciable even in the bulk of the fluid.

In an attempt to understand why this theory works even at rather high Reynolds numbers, one may formally introduce the Kolmogorov internal microscale defined by [24], if the value of ϵ is known. For the first William's case one gets $\eta = 1.8$ mm and for the second $\eta = 1.3$ mm. The ratio of the scale d to η is for both cases close to 25. Apparently if the value of the criterion

$$H_k = d/\eta (\text{Re})^{\frac{1}{2}} \quad [33]$$

is not too large the flow has a laminar or slightly irregular character and our formula [1] may be extended to these conditions,* though one must not expect the similarity of the flow patterns for $Re \gtrsim 1$. The second derivation of our formula for the mean convective velocity where the efficiency γ and the expression for ε are used, is also supporting these arguments because it again requires only the importance of viscosity for shaping flow patterns and not $Re \ll 1$.

However, Williams has published the detailed results only for the two cases. In order to see whether the agreement is not fortuitous I began to think on the possibility of the experimental check of eqn. [1] for enlarged Reynolds numbers. I was not able to find the necessary data in the literature. The discussion of possibilities to undertake a check of the full or partial dependencies in [1] revealed to me that it is quite a job requiring thorough preparation, careful measurements and extensive treatment of data.

On one of those days (mid-April, 1977) I was asked at home to look

* Similar results have been obtained by Golitsyn and Steklov (1977) while determining the height of the turbopause of a planetary atmosphere. For the case of Earth the height of the turbopause, i.e., the level where eddy and molecular diffusivities become equal, is determined as the height where the microscale η reaches about 0.1 of the atmospheric scale height. This was found to be reasonably well for other planets too.

for the soup boiling on the gas stove. Watching for grains and bubbles in the soup, I estimated that their velocities were of the order of several centimeters per second, guessed that the power to the kettle 20 cm in diameter was of the order of several hundreds of watts and looked what formula [1'] was producing. It gave about 5 cm/sec. At that moment, I decided to do the necessary experiments myself right at the kitchen. My hopes were based on the observations of the particles' trajectories. At temperatures before boiling the distinct and rather long parts of the trajectories were almost rectilinear and horizontal. In space and time the parts (and velocities) were rather irregular, which required sufficient statistics.

I realized that I might perform an easy but quantitative check of the linear dependence of the mean velocity on the fluid height d , keeping other parameters constant, i.e., keeping the fire and temperature of the water constant. The experiments took two evenings. At first I had worked out the technology and got preliminary results showing that the dependence $\bar{u} \sim d$ is about right. Then I had received some useful advice at my Institute and in one of the evenings I carried out a series of measurements, the results of which are presented in Fig. 1.

Following is a short description of the "technology" of the experiment which any reader may carry out by himself at his kitchen,

having only a stop-watch and a couple of hours. To measure distances I had drawn by ball-point pen a 1 cm grid at the bottom of a white enameled kettle 20 cm in diameter. The grid did not change in the least during two hours of experiments with water at 80-90°C. The kettle was in a water bath. The bath was composed from a wide frying pan, the kettle standing on three small pieces of a wooden rod 1 cm high and the depth of the water (at 90°C) was about 2 cm. All the construction was on a slow constant gas fire. On a nearby stove was a large teapot with water of about the same temperature on a small fire. The teapot water was used to add water into the pan (evaporation!) and to change its depth in the working kettle. Temperatures were measured by a laboratory mercury thermometer. Inside the kettle the mean water temperature was $83 \pm 1^\circ\text{C}$ for all measurements. It was changing little during measurements and so if one is interested only in the check of the $u \sim d$ dependence one may not have the thermometer but should keep all the fires constant. The depth of the water in the kettle was measured by a ruler. The choice of tracers was a problem at first. Meanwhile I found that almost any dry organic powdered material, e.g., black pepper, may serve as a tracer, because becoming wet it has practically neutral buoyancy. In the experiment described I used powdered (by myself) tea and a dry red wild rose berry (also powdered).

Most of these particles were sitting at the bottom; part were on the surface but some were transported within the water. Their path was measured by an eye on the coordinate grid and the time of the rectilinear parts was measured by the stop-watch. For the results presented in the figure, there were eight layers with a depth from 2 to 10 cm. At each layer there were 35 individual measurements of path and time. During the time of the measurements at each depth a layer of water about 3 mm thick was evaporating and the horizontal scale of the points at the figure reflects this fact.

The dispersion of individual velocity measurements around their means is in the limits of 10-15% which corresponds, I have said, to the usual accuracy of visual measurements. However, an accuracy of each individual measurement seems to be better, therefore these 10-15% reflect evidently a natural dispersion of velocities at convection. Through all the points with bars a direct line could be drawn. Some deviation of the last points might be due to two reasons: (i) an increase of heat losses through the kettle wall, and (ii) an underestimate by eye of the vertical component of the path at larger depths of the water layer.

The value of the proportionality coefficient between $\bar{u}$ and d is about 0.2 sec^{-1} as follows from the data of Figure 1. For $t = 83^\circ\text{C}$

the dynamic viscosity $\mu = 3.5 \cdot 10^{-3}$ P. The values of α and c_p depend for water only very weakly on temperature. Knowing $\bar{u}/d = 0.2 \text{ sec}^{-1}$ we can estimate the heat flux by the formula $f = 72 \mu c_p \bar{u}^2 / \alpha g d^2$. It is 0.2 W/cm^2 . The total flux into the kettle is then about 60 W, which looks a reasonable value.

The knowledge of $\bar{u}$ and f allows one to estimate the basic similarity criteria. Let us do this for the shallowest depth of 2 cm. Then the Reynolds number $Re \approx 300$, the Péclet number $Pe \approx 600$ and the Rayleigh flux number (see eqn. A2.3) $Ra_f \sim 10^6$. The convection is rather irregular though it is not possible to consider the turbulence as fully developed as, e.g., in the atmospheric surface layer. In fact, knowing the heat flux density and the efficiency $\gamma = d/H \approx 10^{-8}$ one can estimate the specific rate of the dissipation $\epsilon = f/\rho H = 10^{-2} \text{ cm}^2 \text{ sec}^{-3}$. Using [24] and [33] one finds $H_k = \eta/d \approx 20$, which seems too small for the existence of a regime of a developed turbulence.

In concluding this section, it is repeated once more that the exact formulation of the conditions of the validity of the formula [1] or [1'] in the sense of the value of the Reynolds number (or the Rayleigh number -- see Appendix 2) is still to be found. Apparently new theoretical and extensive experimental studies have to be performed.

6. Gravitational convection

A number of geophysicists believe that the convection in the Earth's mantle is caused by differentiation of the mantle's material when a heavier fraction is descending and a lighter one is arising (see, e.g., Artyushkov, 1968, Sorokhtin, 1974). How the differentiation is proceeding at a molecular or at some macroscopic level can only be speculated upon. It is believed that this process is taking place at the mantle-liquid core interface. In such an uncertain situation only the simplest phenomenological approach is justified. Such an approach will be tried here.

Equations describing the gravitational or density convection may be written in the Boussinesq-like approximation as follows:

$$\frac{dv}{dt} = - \frac{\nabla p}{\rho_0} + g \frac{\rho'}{\rho_0} + v \Delta v, \quad [34]$$

$$\nabla v = 0, \quad [35]$$

$$\frac{d\rho'}{dt} = \rho_t + k_D \Delta \rho'. \quad [36]$$

Here ρ_0 is a given field of the mean density, ρ' is the deviation of the density from the mean due to the differentiation causing the

convection, ρ_t is the rate of the density differentiation in the volume if such a process is taking place, k_D is a diffusion coefficient of the density deficit. It could probably have a sense similar to the filtration coefficient of a liquid in a porous media (see, e.g., Barenblatt et al., 1972).

The energy equation [6] should be added here together with the equation of state $\rho = \rho(T)$. If the differentiation is taking place at some surface (at the layer's lower boundary, say) then it may be described by a value r_t with the dimension $ML^{-2}T^{-1}$, i.e., of the density flux. At mathematical modelling it is equivalent to the following condition

$$r_t = k_D \partial \rho' / \partial z \quad \text{at } z = 0, \text{ say,} \quad [37]$$

and other boundary conditions are the same as for the thermal convection.

The total flux of the density deficit M at a level z consists of two parts:

$$M(z) = r_t + \int_0^z \rho_t(z) dz . \quad [38]$$

The flux ρ_t plays the role analogous to the heat flux in the thermal convection. To make the analogy quite clear, note that the heat flux

can be connected with the density deficit flux M in the following way:

$$M = \overline{\rho'w'} = \alpha \rho_0 \overline{T'w'} = \alpha c_p^{-1} f, \quad [39]$$

where the overbar means some average. The setting of this relation allows one to write down at once the formula for the mean velocity of the gravitational convection, substituting in [1] M for $\alpha f/c_p$:

$$u \sim \frac{g M d^2}{\rho \nu}^{\frac{1}{2}} = \frac{g M^{\frac{1}{2}}}{\mu} d. \quad [40]$$

The specification of a numerical coefficient in this formula can be done by analysing the energetics of the process as for the thermal convection. The flow with the density deficit flux M rising to the height d in the gravity field g is releasing the potential energy with the rate $M g d$ in the unit column of height d (in fact, $g\rho'$ is the force on the unit volume, $g\rho'w$ is the power of the force, $g\rho'wd = Mgd$ is the total power released in the unit column. This power is spent on the rate of generation of the kinetic energy of the convection which should be equal in average to the rate of its dissipation for a steady process. We have already estimated in Section 2 the dissipation of kinetic energy in the unit column for the plane cell as

$$E_1 = \rho_0 \epsilon d \approx \rho_0 d \cdot 32 \bar{u}^2 / d^2 .$$

Equating this expression to $M g d$ one gets

$$\bar{u} \approx \left(\frac{g d^2}{32 \rho_0 \nu} \right)^{\frac{1}{2}} = \left(\frac{g}{32 \mu} \right)^{\frac{1}{2}} d . \quad [41]$$

Here again is evidenced the analogy with the structure of the velocity field for scales less than the viscous microscale. The role of the dissipation rate ϵ here plays the value of $M g / \rho_0$.

Note that from the equality $E_1 = M g d$ it follows at once that the efficiency of the gravitational convection is unity, if $M = \text{const}(z)$ which is right for the differentiation only at the lower boundary.* The substitution into this equality of the density deficit flux for the thermal convection from [39] gives $\gamma_1 = (g d / c_p) f = (d/H) f$. It looks like a simplest derivation of the formula for the efficiency of the thermal convection $\gamma = d/H$.

* If the separation of the material takes place only within the layer then the total power released at the convection will be equal to

$$Q = \int_0^d g M(z) dz = \int_0^d g dz \int_0^z \rho_t(z') dz'$$

For $g = \text{const}$ and $\rho_t = \text{const}$ $Q = \frac{1}{2} M(d) g d$, where $M(d) = \rho_t d$. The efficiency of such a process is twice less than for the differentiation at the lower boundary only. A general case may be considered similarly to the efficiency of thermal convection (see Hewitt et al., 1975).

Therefore, at convection the power Mgd is transformed at the end into heat and ensures the heat flux which could be measured at the upper surface. Because in the formation of the observed heat flux purely thermal sources can also take part then a constraint on the flux of the density deficit follows from here.

$$M < f/gd . \quad [42]$$

This inequality is of importance for the consideration of processes in the Earth's interior (see following Section 7).

The formula for the velocity scale [40] can be also obtained by the similarity and dimension arguments. The simplest conditions to get it are for $\rho_t = 0$, $Re \ll 1$ and $k_D \ll \bar{u}d$. The last allows us to neglect the diffusive transport of matter. Then in the vorticity balance equation obtained by taking curl of [34] we will get a single dimensional combination $g/\nu\rho_0 = g/\mu$ with the dimension $L^2T^{-1}M^{-1}$, in the boundary condition [37] there is the flux M of dimension $ML^{-2}T^{-1}$ and there is the vertical scale d . Using these three values one constructs the unique velocity scale which is [40]. The time scale does not again depend on the geometry scale d :

$$\tau = d/V \sim (\mu/gM)^{\frac{1}{2}} \quad [43]$$

For the total kinetic energy of the volume d^3 one can get

$$E = c \frac{gd^5}{2\nu} \quad [44]$$

where again $c \approx 1/32$ or $1/72$ depending on the two- or three-dimensionality of the convection. It is evident that this formula is similar to the corresponding formula [20] for the thermal convection.

The found analogy allows one to model the gravitational convection using the thermal one. Then the role of the Prandtl number $Pr = \nu/k$ would play the Schmidt number $Sc = \nu/d_D$ and the role of the Rayleigh flux number defined as

$$Ra_f = \frac{\alpha g f d^4}{\rho_o c_p k^2 \nu} \quad [45]$$

would play the number

$$Ra_m = \frac{gd^4}{k_D^2 \nu \rho_o} = \frac{gd^4}{k_D^2 \mu} \quad [46]$$

A simple example of gravitational convection from everyday experience is the motion in a liquid where gaseous bubbles are forming, such as in the glass with gasified mineral water or with any other bubbling liquid. However, the visualization of these motions and

devising any quantitative measurements proved to be a hard problem for me. After many trials and wasting many bottles of the mineral water, I finally was able to invent a simple and quick experiment based on the check of eqn. [43], i.e., the independence of convection time scale on the depth of the liquid.

As the working fluid I used the mineral water, "Moskovskaya" (from a drill hole within Moscow City), which was poured into a transparent glass flask of a parallelepiped shape with sizes 95 x 79 x 37 mm. It had been noted that adding small particles (of powdered black pepper*) increase strongly the bubble formation and the intensity of motions. The marked release of gas takes place in these conditions for several hours. The experiment consists of the following procedures. First I put into the flask by a pipet the black pepper in water (about 0.5 cm³). Then the mineral water was poured into the flask first up to the line of 2 cm. All this was allowed to settle for a few minutes to cease the motions caused by pouring. Then to the surface of the water a droplet of dye (alcohol solution of brilliant green) was introduced by the pipet. In the water

* Originally the powdered pepper was thought to be used as a tracer; however, after some observations I noticed that the particles' upward motion was caused mainly by gas bubbles sitting on the particles or caught by them and the downward motion originated as from an elastic strike of the rising particles on the surface film of the water.

the droplet formed at once a little cloud from which dye threads or wisps were pulled out. For control such a droplet was introduced into an ordinary water where it spread much slower and the dye remained mainly in the upper layer of the water. Two typical times were measured, the time T_1 , when a dye thread first touched the flask bottom, and the time T_m , when many threads spaced more or less uniformly were touching the bottom. If the first time may be determined more or less distinctly, the second time, T_m , is determined rather subjectively. The results of two series of such measurements are presented in the following table.

TABLE

d, cm	T_1 , sec		T_m , sec	
2	12	16	25	35
3	15	--	25	--
4	--	11	--	30
5	14	--	25	--
6	--	15	--	30
7	12	16	30	30
8	--	14	--	25
9	15	15	30	25

The data of the table show that both times may be considered as independent on the depth of layer of mineral water. The mean velocities of motions calculated from the data of the table change from about 1.4 mm/sec to 6 mm/sec and are presented in a lower part of Figure 1.

For the shallowest layer the Reynolds number is about 30 ($\nu \approx 0.01 \text{ cm}^2/\text{sec}$), and for the deepest one it is about 500. Though these numbers markedly exceed unity, the experience obtained in studying the thermal convection (see Section 5) tells us that here some statistical regularities valid for $\text{Re} \ll 1$ may still be operative. Therefore, the absence in our measurements of any systematic dependence of the characteristic times of motions on the layer depth may be considered as an argument in favor of the theory proposed here.

7. Applications to the mantle convection

Here a brief discussion will be presented of the developed theory to the Earth's mantle. Because here (following McKenzie et al., 1974) parameters of the medium are constant, then it may be considered only as the first rough approximation of the convection in the mantle. In reality the viscosity depends strongly on the temperature and pressure (see Carter, 1976) which exerts important influence on the flow pattern. In particular the computations of the thermal convection by Houston and DeBremaecker (1975), carried out with the Herring-Nabarro viscosity depending exponentially on temperature, give an appreciable intensification of the convection in the regions of decreased viscosity and its attenuation there where the viscosity is large. Nevertheless these computations and experiments by Booker (1976) show that the flow patterns and intensity of the motion are

not too drastically different from the convection with constant parameters of the material. Therefore our formulas are giving right orders of convective velocities if one uses some effective viscosity value. However, this question requires additional studies, laboratory and numerical ones.

The values of thermal convection velocities of order 1 cm/yr obtained here (after McKenzie et al., 1974) seem to be insufficient, because many lithospheric plates are moving several times faster. If one takes into account that the plates are moving as a whole, dragging each other, the oceanic plates are diving under continental ones, then one would feel safer if the mantle's motions would have velocities, say, of order 10 cm/yr.

The structure of the formula [1] shows that this may be reached by an increase of the coefficient of thermal expansion α and/or by a decrease of the dynamic viscosity μ . Hewitt et al. (1975) note that the value of α is rather uncertain and could be, in principle, increased by an order of magnitude which would increase the velocities by a factor of 3. However, this would also mean that the efficiency $\gamma \sim d/H \sim 1$. But at $d \sim H$ some other complexities arise (see their discussion by Hewitt et al., 1975) leading to possible inapplicability of our simple considerations. One should not exclude that the value of viscosity $\nu = 2 \cdot 10^{-17} \text{ m}^2 \text{ sec}^{-1}$ (or $\mu = 7.4 \cdot 10^{21} \text{ P}$) adopted by

McKenzie et al. (1974) and here is also considerably overestimated (see also Carter, 1976). So it looks like it is possible to increase $\bar{u}$ up to 10 cm/yr even for the purely thermal convection in the upper mantle.

Let us discuss now the possible gravitational convection. Here the whole mantle, including the lower one, will be considered, i.e., the layer of some 3000 km thickness, due to the belief that the differentiation goes on the mantle-core interface. Not going into details of the geophysics of these questions, we present here only some estimates of the intensity of such a convection and point out some constraints.

Consider first inequality [42]: $M < f/gd$. Let $f = 0.06 \text{ W/m}^2$ and $d = 3000 \text{ km} = 3 \cdot 10^6 \text{ m}$. Then $M < 2 \cdot 10^{-9} \text{ kg/m}^2\text{sec} = 6 \cdot 10^{-2} \text{ kg/m}^2\text{yr}$. For the period of time T_0 the density of the mantle material will be changed by

$$\Delta\rho \sim r_t T_0/d = M T_0/d \lesssim f T_0/gd^2. \quad [47]$$

We neglect further the non-uniformity of the differentiation rate in the process of the Earth's evolution. Due to the models of the evolution by Keonjan and Monin (1975) and Monin and Keonjan (1976) this rate for the last four billion years changes less than

twice. Then taking the present value of f and $T_0 = 4$ Aeons, one gets from [47] that $\Delta\rho \lesssim 100 \text{ kg/m}^3 = 0.1 \text{ g/cm}^3$. This value could be somewhat increased if one assumes that some part of the heat released at gravitational convection goes to the support of the differentiation reactions which are, evidently, endothermic. Then instead of [42] we should write

$$M \text{ gd} < f + Q, \quad [48]$$

where Q is the heat spent in a unit column for the support of the reactions. Nevertheless, the value $\Delta\rho \sim 1 \text{ g cm}^{-3}$ adopted by Artyushkov (1968) and Sorokhtin (1974) seems to be too high not only from the point of view of the constraints [42] or [48], but also regarding estimates of the energy released at the gravitational differentiation. In fact, due to Monin and Keonjan (1976) and several other models the total energy released at the process is of order $1.5 \cdot 10^{31} \text{ J}$ for the whole Earth's history. If all this energy would be brought up to the surface uniformly, then the geothermal flux would be of the order 0.2 Wm^{-2} , i.e., thrice the present value. The excessive energy can go only to the heating of the core. If the core mass is of the order 10^{25} kg then the heating for 4 Aeons would be about 2000 K. It would be less if part of the energy would go for the support of the differentiation reactions.

For illustrative purposes we present now estimates of mean velocities for the gravitational convection. Let $M \sim 2 \cdot 10^{-9}$ kg/m²sec. For the dynamic viscosity of the (lower) mantle $\mu \sim 10^{27}$ kg/m sec = 10^{26} P (see McKenzie et al., 1974) one gets from [41] that $\bar{u} \sim 0.1$ cm/yr. Monin and Keonjan (1977) assumed the value of μ by three orders less, as a representative for the whole mantle. Then $\bar{u} \sim 3$ cm/yr. Not being a specialist on the Earth's interior geophysics, I end the discussion at this point.

8. An attempt to classify geophysical flows

Discussing theories developed here, we have already referred to the similarity theory for circulation of planetary atmospheres and to Kolmogorov's theory of turbulence in the viscous range of scales. The last is found to be the direct analog of the viscous fluid convection studied here. Therefore, we may consider that the thermal and gravitational convection at the not too large Reynolds numbers and turbulence in the dissipation range are forming a family of forced viscosity dominated flows.

However, there are more common features among all the aforementioned kinds of flow. The first is that the total kinetic energy of a general circulation and of convection does not depend on the total mass of the atmosphere or of the convecting fluid.

For convection this property is evident from eqn. [20] for the thermal case and eqn. [44] for the gravitational case. The kinetic energy of the circulation of a planetary atmosphere is equal (up to a multiplier or a non-dimensional function of angular velocity) to

$$E \sim 2\pi\sigma^{1/8}c_p^{-1/2}q^{7/8}r^3, \quad [49]$$

where σ is the Stefan-Boltzmann constant, $q = q_0(1-A)/4$ is the mean rate of solar energy reaching the planet accounting for its albedo A , r is the planetary radius (see Golitsyn, 1970, 1973). Independence of the kinetic energies on the mass of flows is a more general feature which might be used for a classification.

However, many types of flow do not have this property, but nevertheless, they have another more general feature which we shall discuss now, starting from circulations. The eqn. [49] after some simple transformations can be written as

$$E = \frac{(\kappa-1)^{1/2}}{2} Q \cdot \frac{r}{c_e} \quad [49']$$

where $\kappa = c_p/c_v$, $Q = 4\pi r^2 q$ - the total power of the solar energy assimilated by a planet, $c_e = [(-1)c_p T_e]^{1/2}$ is the sound velocity, $T_e = (q/\sigma)^{1/4}$ is the temperature of the equilibrium radiation of the planet. The quantity $\tau_e = r/c_e$ is the time for propagation of

a perturbation in the atmosphere in global scale. As is known (Landau and Lifshitz, 1959, §48) the sound velocity is characteristic for reaching the local thermodynamic equilibrium. Therefore up to a multiplier of order unity the total kinetic energy of circulation is

$$E \sim Q_e \tau, \quad [49'']$$

i.e., it is equal to the total power of the radiation assimilated by a planet times the time of perturbation relaxation in the global scale.

Expression [44] for the total kinetic energy of the gravitational convection $E \approx M g d^5 / \nu$ has exactly the same structure. The combination $q_g = M g d$ is the power of this convection in a unit column. The total power of the convection in the whole volume (or the rate of potential energy release in the volume) is $Q_g = q_g d^2 = M d g^3$. From the other hand, $d^2 / \nu = \tau_\nu$ is the viscous relaxation time in the volume with characteristic size d . As a result

$$E_g \approx M g d^5 / \nu = Q_g \cdot \tau_\nu \quad [44']$$

The peculiarity of the thermal convection is that only some part of the heat power brought in a fluid is spent on the generation of kinetic energy of convective motions. The part determined by

the efficiency coefficient of the convection is equal to $\gamma = \alpha g d / c_p = d/H$. Accounting for this circumstance, eqn. [20] can be rewritten as

$$E_r = \frac{d}{H} \frac{f d^4}{\nu} = \gamma F t_v, \quad [20']$$

where $F = f d^2$ is the total heat flux introduced into the fluid. The same structure is having eqn. [27] for the kinetic energy of a volume of turbulent fluid with size $d < \eta$.

The similar form may be given also to the expression for the kinetic energy of a volume of locally isotropic and homogeneous turbulent flow relative to its closest volumes with the size of $\eta < d < L_e$, where L_e is the turbulence external scale, i.e., for d in the inertial interval (Kolmogorov, 1941). The energy of such a volume with the mass $M = \rho d^3$ is

$$E \approx M \epsilon^{2/3} d^{2/3} = Q d / (\epsilon d)^{1/3}, \quad [50]$$

where $Q = M \epsilon$ is the total energy rate brought into the volume. Because in the inertial range of turbulence $(\epsilon d)^{1/3} \approx V$ the relative rms velocity for two points separated by the distance d then

$$E \approx Q \cdot d / V \approx Q \tau_v, \quad [50']$$

where τ_v is the characteristic lifetime of eddies with scale d .

Note, however, that the locally isotropic and homogeneous flow in the inertial range does not belong to the class of flows whose kinetic energy (in our relative sense) does not depend on the fluid mass. Because $\epsilon = Q/M$ the relative kinetic energy, as follows from [50], is proportional to $M^{-1/3}$. Analogous dependence of the total kinetic energy from mass has been obtained in some models of general circulation, considered as a large scale convection on non-rotating planets (Gierasch et al., 1970, Leovy and Pollack, 1973, Burangulov and Zilitinkevich, 1976). In these models one may also obtain formulas of the type of [50]. In Appendix 3 we consider a turbulent convection of a fluid in a bounded region. This kind of convection also belongs to this type of flow. Together with the aforementioned models of circulation and turbulent flow in the inertial range it forms a family of flows with mean velocities proportional to $(\epsilon d)^{1/3}$. This group of flows may be called a family of forced turbulence dominated flows or inertial, because the inertial non-linearity and resulting turbulent mixing determine their structure and intensity.

We see that for quite a number of forced geophysical flows their total kinetic energy is determined by the product of total power brought into the fluid and of characteristic relaxation time.

Note that in all the cases considered here this time is always the smallest from all the times which can be constructed from external parameters in the equations. It is true, however, that this smallest time is usually only one if we believe in the validity of corresponding self-similarity hypotheses neglecting various external parameters. This allows one to propose the following approximate rule which could be called a "principle of the fastest response". The "principle" says that the kinetic energy of a forced steady flow of a fluid system is of the order of total power brought into the system times the shortest relaxation time characteristic to the system.

If one is not using the similarity theory then this "principle" allows one to write at once the expression for the total kinetic energy of motions. It was this "principle" noted by me for the general circulation which has been used in 1970 (Golitsyn, unpublished) for obtaining an expression of the kind of eqn. [20], but without accounting for the convection efficiency γ (it looks like nobody at the time had any idea of it). I was getting then $V \sim 30$ cm/yr for the mantle and the excess was mainly not due to not accounting for efficiency but due to not realizing that the numerical constant c in [20] should be about $1/30$ (or even less).

We see that obedience of the flows considered here to the "principle of the fastest response" is their most general property.

However, to drive away an impression on the universality of the principle I want to present an example of a system where it gives, at best, an estimate from below of the kinetic energy. This is the circulation in atmospheres of large and fast rotating planets Jupiter and Saturn. A detailed discussion of their circulation was given by Golitsyn (1970, 1973). It looks like the fast rotation is a factor strongly stabilizing large-scale motions and not allowing the system to relax by the fastest way.

Results of this section are mainly of methodical character. However, it seems that the "principle of the fastest response" may also have a heuristical value as it has had for this paper for which it served as a first impulse.

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problem after six years was obtained through reading the paper by Hide (1976) where there was a reference to McKenzie et al. (1974). I am deeply grateful to R. Hide for sending me a preprint of his paper. Of much use and importance for me were several reprints which D. P. McKenzie had sent to me and I appreciate very much his quick and kind reaction to my questions.

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APPENDIX 1

SOME GENERAL FORMULAS TAKING INTO ACCOUNT THE EFFICIENCY OF CONVECTION

The concept of a convection efficiency measured by the coefficient $\gamma = d/H = \alpha g d / c_p$, suggested by Lliboutry (1972) and by Hewitt et al. (1975) seems to be an important achievement of hydrodynamics the use of which has not yet spread wide enough. In this connection it should be noted that the quantity γ in an implicit form enters in fact many important formulas and definitions of hydrodynamics. To help clarify this concept I wish to rearrange several formulas into a form where the efficiency γ will be present explicitly.

First of all two formulas where there is a temperature difference ΔT . Introduce the entalpy difference $\Delta e = c_p \Delta T$. Then the Rayleigh number can be transformed as follows:

$$Ra = \frac{\alpha g \Delta T d^3}{\kappa \nu} = \frac{d}{H} \frac{\Delta e d^2}{\kappa \nu} = \gamma \frac{d^2 e}{\kappa \nu} \quad [A.1.1]$$

Similarly, the thermal Rossby number is transformed:

$$Ro_T = \frac{\alpha g \Delta T d}{4 \Omega^2 b^2} = \gamma \frac{\Delta e}{4 \Omega^2 b^2} \quad [A.1.2]$$

where Ω is the angular velocity of rotation of a vessel with a lateral scale b . Convection there from the point of view close

to the one used here has been consider by the author (1977b).

Finally, the non-dimensional height in the Monin-Oboukhov similarity theory of the stratified turbulent boundary layer (see Monin and Yaglom, 1965, §7).

$$\zeta = \frac{z}{L} = \frac{z \alpha g f}{\rho c_p u_*^3} = \gamma(z) \frac{K f}{\rho u_*^3} \quad [\text{A.1.3}]$$

where $\gamma(z) = z/H$ and u_*^z is the friction velocity. Though γ does not play here the role of the efficiency, its presence in these formulas seems to be instructive.

APPENDIX 2

RELATION OF THE DEVELOPED ELEMENTARY THEORY OF CONVECTION WITH USUAL REPRESENTATIONS

Usual similarity criteria for the thermal convection in the existing literature are the Rayleigh and Prandtl numbers. The first of these have not yet been present in the theory here. As we will see later, this happened because we were not considering the structure of thermal boundary layers. Let us show how the Rayleigh number is emerging here and how it is connected with the similarity criteria which have already been introduced.

In the non-dimensional system (15')-(17') there are three similarity criteria: (i) the Reynolds number

$$Re = \frac{ud}{\nu} = \frac{\tau_v}{\tau_v} = \left(\frac{f}{\mu H} \right)^{\frac{1}{2}} \frac{d^2}{\nu}, \quad [A.2.1]$$

(ii) the Péclet number

$$Pe = \frac{ud}{k} = \left(\frac{f}{\mu H} \right)^{\frac{1}{2}} \frac{d^2}{k} \quad [A.2.2]$$

and (iii) the efficiency coefficient γ . At the same time from the dimensional parameters present in [15]-[17] one may construct the Rayleigh number defined by the heat flux:

$$Ra_f = \frac{\alpha g f d^4}{\rho c_p k^2 \nu} = \gamma \frac{f d^3}{\mu k^2} = \frac{f d^4}{k^2 \mu H} . \quad [A.2.3]$$

Comparing this expression with [A.2.1] and [A.2.2] we get

$$Re = Ra_f^{1/2} / Pr , \quad [A.2.4]$$

$$Pe = Ra_f^{1/2} . \quad [A.2.5]$$

To connect the Rayleigh flux number with an ordinary Rayleigh number [A.1.1] one has to know the dependence of the Nusselt number (see eqn. [30]) on the Rayleigh number. The study of the dependence $Nu(Ra)$ is the usual goal in the laboratory convection experiments. It may be also obtained from theoretical considerations of the energy and vorticity balance in the boundary layer (see Turcotte and Oxburgh, 1967, McKenzie et al., 1974). The measurements and the considerations produce the dependence

$$Nu \approx \beta Ra^{1/3} , \quad [A.2.6]$$

where β is some numerical coefficient (about 0.1-0.2 in dependence on the problem conditions). Using [A.2.3], [A.1.1] and [A.2.6] one gets

$$Ra_f = \beta Ra^{4/3}, \text{ or } Ra = (Ra_f / \beta)^{3/4} . \quad [A.2.7]$$

Comparing [1'] with [A.2.3] or [A.2.7] we may obtain

$$u \approx \left(\frac{Ra_f}{32} \right)^{1/2} \frac{k}{d} = \left(\frac{\beta}{32} \right)^{1/2} Ra^{2/3} \frac{k}{d} \quad [A.2.8]$$

Expressions of this kind but without the numerical coefficient have been obtained by Turcotte and Oxburgh (1967) and McKenzie et al. (1974). Using the Rayleigh flux number one can express also the temperature difference in the main body of the fluid:

$$\delta T \approx \frac{f}{\rho c_p V} \approx (32 Ra_f)^{1/2} \frac{k\nu}{\alpha g d^3} \quad [A.2.9]$$

Related to the total temperature difference ΔT causing the convection the difference δT is equal to

$$\frac{\delta T}{\Delta T} \approx \frac{(32 Ra_f)^{1/2}}{Ra} \approx (32\beta)^{1/2} Ra^{-1/3} \approx 32^{1/2} \beta^{-3/4} Ra_f^{-1/4} \quad [A.2.10]$$

Therefore with the increase of the Rayleigh number the temperature profile would appear more and more like isothermal one in the bulk of fluid comparing with temperature drops in the thermal boundary layers. Laboratory experiments and numerical computations (see, e.g., McKenzie et al., 1974) show that for a developed convection the main temperature changes are in the boundary layers of about $\frac{1}{2}\Delta T$ in each (upper and

lower ones). Let us define the thickness of the boundary layers δ using the boundary condition [17] in such a way that

$$f = \frac{k\Delta T \rho c_p}{2\delta} \quad [\text{A.2.11}]$$

Comparing this expression with [30] we get

$$\text{Nu} = d/2\delta ,$$

from where accounting for [A.2.6], it follows that

$$\delta \sim (\frac{1}{2}\beta)d \text{ Ra}^{-1/3} = \frac{1}{2}\beta^{-3/4} d \text{ Ra}_f^{-1/4} \quad [\text{A.2.12}]$$

If at the lower boundary we know the heat flux but the total temperature change ΔT is not known, we may estimate it from [A.2.9] taking into account [A.2.12] as

$$\Delta T = \frac{2f\delta}{\rho c_p k} = \left(\frac{\text{Ra}_f}{\beta} \right)^{3/4} \frac{kv}{\alpha g d^3} . \quad [\text{A.2.13}]$$

The expressions of the type of [A.2.12] and [A.2.13] are also in agreement in McKenzie et al. (1974) and results of their computations agree well with the dependencies $\delta \sim f^{-1/4}$ and $\Delta T \sim f^{3/4}$ following from the expressions.

The last thing which is useful to estimate is the ratio of temperature gradients in the main interior of the fluid -- $\nabla_i T$ and in the boundary layers -- $\nabla_\delta T$. It follows from [A.2.10], [A.2.12] and [A.2.13] that

$$\frac{\nabla_i T}{\nabla_\delta T} \approx \frac{32^{\frac{1}{2}}}{\beta^{3/4}} Ra_f^{-\frac{1}{4}} \frac{\delta}{d} \approx \left(\frac{2}{\beta}\right)^{3/2} Ra_f^{-\frac{1}{2}} \quad [A.2.14]$$

APPENDIX 3

TURBULENT CONVECTION

At last we consider the turbulent convection when the Reynolds number is very large and the direct role of viscosity in the bulk of the fluid is not important. Exact conditions for the existence of such a regime in terms of external parameters or similarity criteria have yet to be established mainly experimentally, in the laboratory or by computations. Though this regime comes off the frame of the basic subject of this paper, nevertheless its consideration is justified here from the point of view of Section 8, because the regime as we will see later is the closest analog of the regime of developed turbulence, described by the Kolmogorov-Oboukhov theory. This adds to our scheme of classification of flows by one more object.

Based on the vorticity equation [15] we see that for a steady case the main balance is between inertial and buoyancy terms. Estimating the entalpy gradient from the energy equation [16] and substituting it into [15] we obtain the following estimates of scales of velocity and entalpy gradient (again at the main body of fluid).

$$u = \left(\gamma \frac{f}{\rho} \right)^{1/3} = \left(\frac{f d}{\rho H} \right)^{1/3} = \left(\frac{\alpha g f d}{\rho c_p} \right)^{1/3} \quad [A.3.1]$$

$$\nabla e = \frac{1}{\gamma^{1/3} d} \left(\frac{f}{\rho} \right)^{2/3} \quad [A.3.2]$$

From the last formula an expression follows for the temperature

difference in the bulk of fluid with height of order d:

$$\delta T \approx \frac{1}{\gamma^{1/3} c_p} \left(\frac{f}{\rho} \right)^{2/3}. \quad [A.3.3]$$

The expressions [A.3.1] and [A.3.3] will be identical to the expressions for variances of velocity and temperature for the free convection in the atmosphere if one substitutes there the running height z for the layer depth d and remember that for a gas $\alpha = 1/T$, where T is a characteristic temperature. Let us remember that the expression for the velocity variance in atmospheric convection has first been obtained by Prandtl (1932), and the expression for the temperature variance has been obtained by Oboukhov (1960), who being unaware of the Prandtl paper has also obtained an expression of the [A.3.1] type and several other useful expressions. Both these scientists were using similarity and dimensional arguments and here the scaling analysis is used.

Because here eqn. [A.3.1] could be written as $u \sim (\epsilon d)^{1/3}$. This clarifies the analogy between regimes of turbulent convection and of locally isotropic and homogenous turbulence. We see that Prandtl (1932), nine years before Kolmogorov (1941) was not too far from discovering the laws of local structure of turbulence.

If we use the scales u and $\nabla \epsilon$ for non-dimensionalizing the equation system [15]-[17], then we will have as the similarity

criteria the Reynolds and Péclet numbers

$$Re = \frac{ud}{\nu} = \left(\frac{fd}{\rho H} \right)^{1/3} \frac{d}{\nu}, \quad [A.3.4]$$

$$Re = \frac{ud}{k} = \left(\frac{fd}{\rho H} \right)^{1/3} \frac{d}{k} = RePr \quad [A.3.5]$$

As another similarity criterion for convective flows we may again take the Rayleigh flux number defined by [A.2.3]. Here its relations with the other criteria differ from the previous case and are of the form

$$Re = Ra_f^{1/3} Pr^{-2/3}, \quad [A.3.6]$$

$$Pe = Ra_f^{1/3} Pr^{1/3}. \quad [A.3.7]$$

The temperature difference in the bulk of fluid is in these terms equal to

$$\delta T \sim Ra_f^{2/3} \frac{(k^2 \nu)^{2/3}}{\alpha g d^3}, \quad [A.3.8]$$

and the velocity scale

$$u \sim (Ra_f Pr)^{1/3} \frac{k}{d}. \quad [A.3.9]$$

The structure of the thermal boundary layer remains the same as in the viscous case considered in Appendix 2 because the dependence $Nu \sim Ra^{1/3} \sim Ra_f^{1/4}$ is valid here too. The last formula here is the ratio of temperature gradient in the bulk of fluid and in the boundary layer:

$$\frac{\nabla_i T}{\nabla_\delta T} \approx (Ra_f Pr)^{-1/3} \quad [A.3.10]$$

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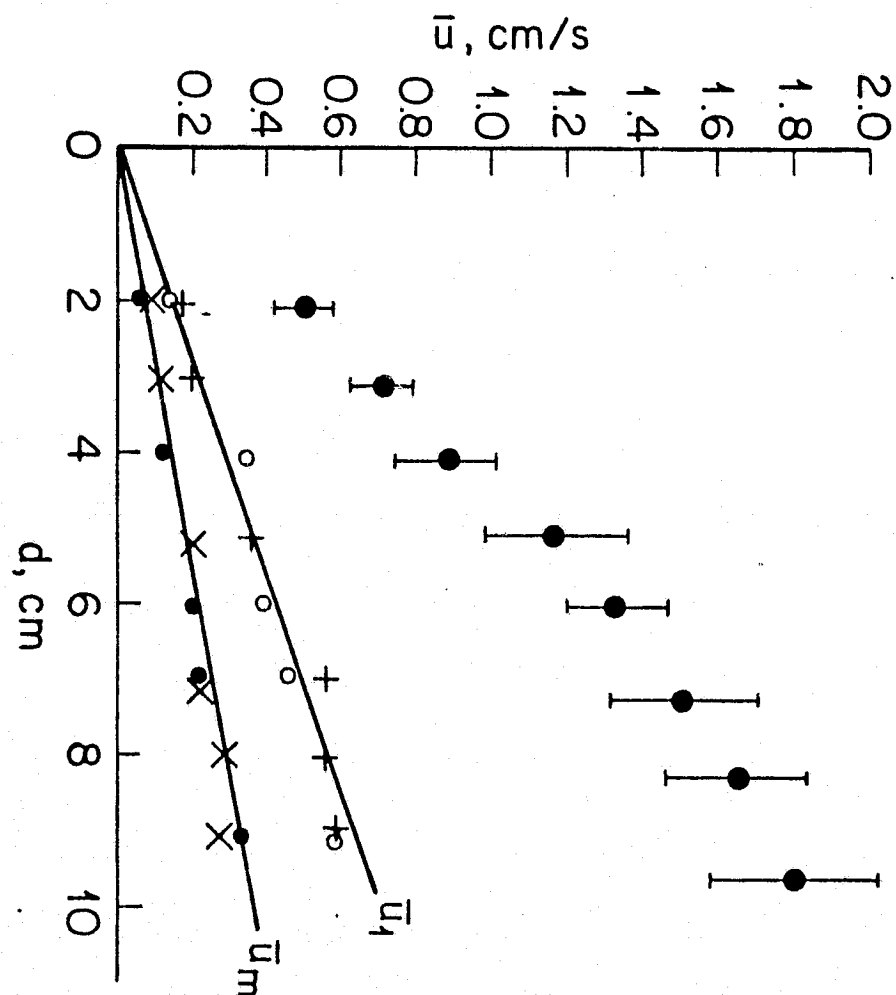


Figure 1

The dependence of rms velocity $\bar{u}$ of the fluid on its depth. Upper solid dots with standard deviations (bars) are for the thermal convection. The lower part is for the gravitational convection. $\bar{u}_1 = d/T_1$ is the maximal velocity, $\bar{u}_m = d/T_m$ is the mean velocity, dots and crosses are for two runs of the experiment according to Table 1 (see also text).